

# Constructing Euclidean Color Spaces based on Color Difference Formulas

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## Abstract

*This paper shows a computational technique of how to construct a nearly isometric transformation from a color space with a non-Euclidean color difference formula into an Euclidean space. The resulting transformation is a combination of a one-dimensional color lookup table (CLUT) to transform lightness values and a two-dimensional CLUT to transform chroma and hue coordinates. As an example the CIEDE2000 formula and a new optimized color difference formula for CIECAM02 was used and a transformation into an Euclidean space was calculated. The mean isometric disagreement was far below 3%. Color tolerance ellipsoids were plotted for both investigated color difference formulas and transformed into the new Euclidean space to illustrate the performance of the method. A movie of how the CIEDE2000 system is embedded into an Euclidean space is made available at the website: <http://munsell.cis.rit.edu/~pmupci/>*

## Introduction

A perceptually uniform color space, in which Euclidean distances highly agree with perceptual color differences, is desired in many fields of imaging science, including color image compression, device gamut mapping and color engineering. In recent years much effort led to the standardization of new color spaces like CIELAB or color appearance spaces like CIECAM02 [1]. Unfortunately, the perceptual uniformity of these spaces is not sufficient for various applications so that new color difference formulas were developed and standardized, such as CMC [2], CIE94 [3] and CIEDE2000 [4, 5]. Visual experiments show that the CIECAM02 space is also not perceptually uniform and new color difference formulas can be applied to enhance the correlation to the visual data. These color difference formulae are only reasonable defined for relatively small color differences, 0-5  $\Delta E_{ab}^*$  [6]. To calculate larger distances the geodesics have to be calculated [7][8] that is complicated since the Euler-Lagrange differential equation must be solved. If an isometric transformation into an Euclidean space can be found, than the calculation becomes much simpler, because the geodesics in Euclidean spaces are straight lines and even large distances can be calculated by calculating the Euclidean metric. Here the effect of *diminishing returns in color difference perception* [9] (i.e. perceived large and medium color differences are smaller than the concatenation of threshold differences along the geodesic between the corresponding colors) is not taken into account.

Unfortunately, an isometric (length preserving) transformation to an Euclidean space according to such a color difference formula is not possible. The reason is the geometrical property

of the space: A necessary condition for the existence of such a isometric transformation is that the Gaussian curvature is zero throughout the space (Theorem Egregium) (see e.g. Wyszecski, Stiles [8], page 658). It can be proven that this condition is not satisfied for each of the formulas mentioned above. Since the Gaussian Curvature of the spaces for each of these color difference formulas is small it is possible to construct a transformation into an Euclidean space that is "nearly" isometric, i.e. that the disagreement between distances calculated by the formulas and Euclidean distances in the new color space is not zero but small enough for most applications. Since the Gaussian curvature is a local property of a space, the disagreement between Euclidean distances in the new space and color differences in the underlying space is in general larger in regions where the Gaussian curvature is large. It should also be mentioned that the statistical variability of the visual data that is the basis of the color difference formulas is quite high and it is possible to construct transformations into Euclidean spaces for that the isometric disagreement is far below the noise level of the visual data.

In this paper we show how to derive an Euclidean color space in high agreement with a color difference formula. Previous approaches used analytical methods to derive closed formulas: for the CMC formula [10], for the CIE94 formula [11, 12] and for the first quadrant of the CIEDE2000 formula [13]. For the whole domain of the complex CIEDE2000 formula an analytical transformation into an Euclidean space could not be found. We present in this article a computational method that leads to a simple color look-up table (CLUT), which transforms a non-Euclidean color space into an Euclidean space with minimal isometric disagreement [14]. As examples we use the CIEDE2000 formula on CIELAB and a new color difference formula defined using CIECAM02.

## New Color Difference Formula for CIECAM02

A color-tolerance dataset was formed that consisted of the RIT-DuPont 156 color-difference pairs [15] and the Qiao, et al. 44 hue-difference pairs [16]. CIECAM02 JCh coordinates were calculated for D65, 1500 lx, background reflectance  $Y_b = 100$ , and an average surround. These 200 difference pairs all had the identical visual difference, equivalent to a  $\Delta E_{ab}^*$  of unity for a near neutral centered at  $L^*$  of 50. Nonlinear optimization was performed to derive a new formula where the coefficient of variation was minimized. The equation is listed below. For this dataset, this simple equation had equivalent performance to CIEDE2000. (These data

were also used to derive CIEDE2000.)

$$\Delta E_{02-OPT} = \left[ \left( \frac{\Delta J}{k_J S_J} \right)^2 + \left( \frac{\Delta C}{k_C S_C} \right)^2 + \left( \frac{\Delta H}{k_H S_H} \right)^2 \right]^{1/2} \quad (1)$$

$$S_J = 0.5 + (\bar{J}/100)^2 \quad (2)$$

$$S_C = 1 + 0.02\bar{C}$$

$$S_H = 1 + 0.01\bar{C}$$

$$k_J = k_C = k_H = 1$$

## Computational Euclideanization

Lightness values can be considered separately from chroma and hue values, because lightness differences are independent of chroma and hue for each of the color difference formulas mentioned in the introduction. For this reason the lightness values in the Euclidean space can be simply calculated by integrating the color difference formula along the lightness axis.

### Calculating the Lightness of the new Euclidean Space

For the CIEDE2000 color difference formula the lightness  $L_{00}^*$  in the new euclidean color space is defined by the following integral :

$$L_{00}^*(L^*) = \int_0^{L^*} \frac{dt}{k_L S_L(t)} \quad (3)$$

where

$$S_L(t) = 1 + \frac{0.015(t-50)^2}{\sqrt{20+(t-50)^2}}. \quad (4)$$

This integral can be solved numerically for a large number of equidistant  $L^*$  values,  $0 = L_1^* \leq \dots \leq L_n^* = 100$ . The result is a list of  $n$  lightness pairs  $[L_1^*, L_{00}^*(L_1^*)], \dots, [L_n^*, L_{00}^*(L_n^*)]$  that can be used as a one dimensional CLUT to calculate the new lightness values. Intermediate lightness values can be calculated by means of linear interpolation. Figure 1a shows the lightness transformation for  $k_L = 1$ .

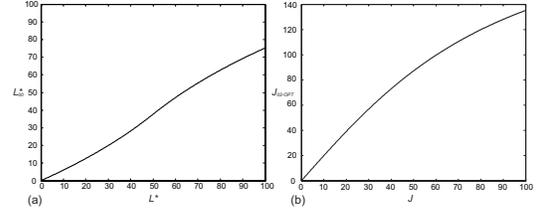
For the CIECAM02 space with the new optimized color difference formula the following integral have to be solved:

$$J_{02-OPT}^*(J) = \int_0^J \frac{dt}{k_J S_J(t)} \quad (5)$$

where  $S_J$  is defined in equation (2). Here a one-dimensional CLUT can be constructed in the same way as for the CIEDE2000 formula. Figure 1b shows the lightness transformation for  $k_L = 1$ .

### Calculating Lightness Independent Coordinates of the new Euclidean Space

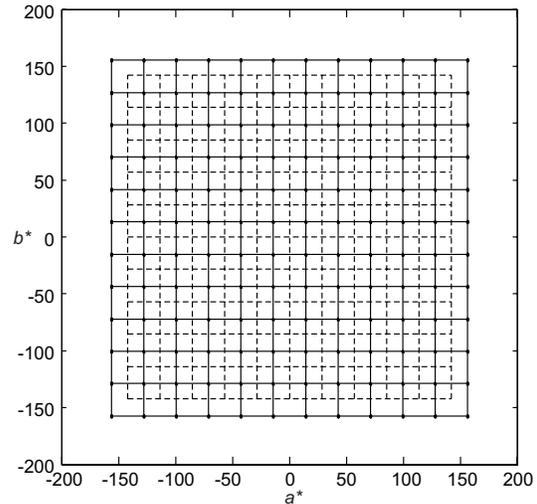
The calculation of chroma and hue is much more complex, since hue differences are dependent of chroma. Particularly for CIEDE2000, an analytical solution is difficult because the rotational term depends on chroma and hue. To construct a reasonable, nearly isometric, analytical transformation is therefore extremely difficult. We present here a computational method based on a multigrid optimization.



**Figure 1.** Lightness transformation functions a) for the CIELAB color space with the CIEDE2000 color distance formula ( $k_L = 1$ ). b) for the CIECAM02 space with the new optimized color distance formula ( $k_L = 1$ )

### The Multigrid Optimization

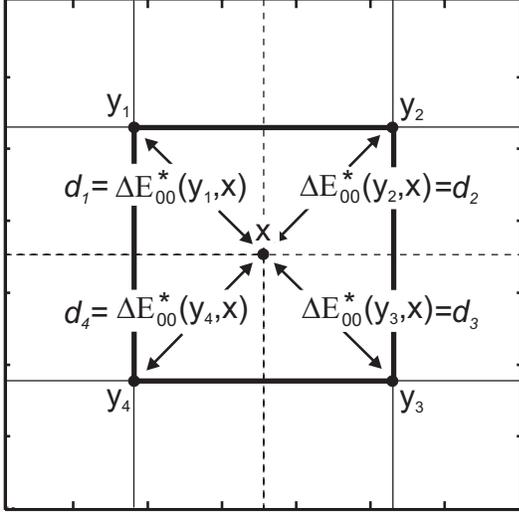
We start with two rectangular grids covering the whole plane of constant lightness with their vertices. For better understanding we denote one grid as  $G_1$  and the other grid as  $G_2$ . The grids are constructed in a way that each mesh of grid  $G_1$  encloses exactly one vertex of grid  $G_2$ , and each mesh of grid  $G_2$  encloses one vertex of grid  $G_1$ . Figure 2 shows two starting grids in the  $a^*, b^*$  plane of CIELAB space.



**Figure 2.** Two starting grids of the multigrid optimization covering the whole plane of constant lightness with its vertices.

In a preliminary step the distances according to the investigated color difference formula are calculated between the mesh vertices of grid  $G_1$  and the vertex of grid  $G_2$  enclosed by this mesh. This has to be done for each mesh of grid  $G_1$  enclosing a vertex of grid  $G_2$ . Figure 3 shows this for one mesh and the CIEDE2000 color distance formula. Additionally, the distances between the mesh vertices of grid  $G_2$  and the enclosed vertex of grid  $G_1$  are calculated using the investigated color difference formula. This has to be done also for each mesh of grid  $G_2$  enclosing a vertex of grid  $G_1$ . For each mesh of both grids we store the four distances, which remain unchanged for the rest of the calculation.

The idea of the multigrid optimization is to move the vertex enclosed by a mesh in order to minimize the difference of the distances calculated in the preliminary step and the corresponding euclidean distances. This can be done for each vertex of grid  $G_1$



**Figure 3.** Calculating the  $\Delta E_{00}^*$ -distances between the vertices of a mesh of grid  $G_1$  (solid lines) and the enclosed vertex of grid  $G_2$  (dashed lines) in a preliminary step.

enclosed by a mesh of grid  $G_2$ . The result is a new grid  $G_1$ . Each mesh of the new grid  $G_1$  enclose one vertex of grid  $G_2$  and we can relocate this vertex in order to minimize again the difference of distances calculated in the preliminary step and the corresponding euclidean distances. This can be done for each vertex of grid  $G_2$  enclosed by a mesh of the new grid  $G_1$ . The result is a new grid  $G_2$ . This procedure can be continued resulting in an iteration that minimizes in each mesh of both grids the difference of distances calculated by the investigated color difference formula in the preliminary step and the corresponding euclidean distances.

Mathematically, we have to solve for each mesh an optimization problem that is implicitly coupled with the optimization problems of the neighboring meshes. Each optimization problem has the following structure

$$F(x) = \sum_{i=1}^4 (d_i - \|x - y_i\|_2)^2 = \min \quad (6)$$

where  $d_i$ ,  $i = 1, \dots, 4$  are the distances calculated using the investigated color difference formula in the preliminary step (see Figure 3) and  $x$  is the vertex of one grid enclosed by the mesh vertices  $y_i$ ,  $i = 1, \dots, 4$  of the other grid.

Since the optimization problems for each grid are coupled only one iteration step of a gradient descent method to solve problem (6) is performed in each step of the multigrid optimization. We used the steepest descent as the gradient descent method, i.e., vertex  $x$  is relocated as follows:

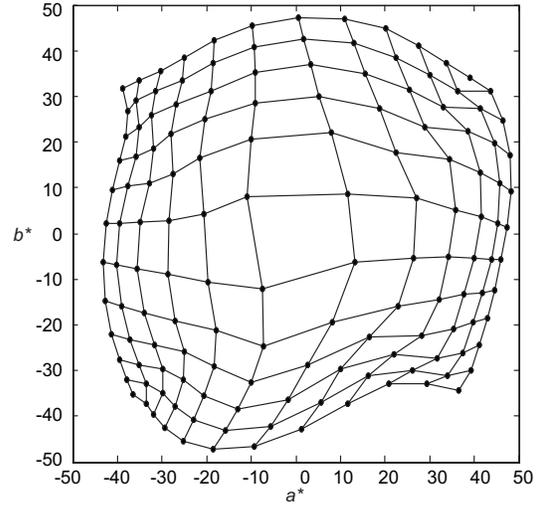
$$x_{\text{new}} = x - s_x \cdot \nabla F(x) \quad (7)$$

where  $s_x > 0$  is a appropriate step length that ensures that the vertex is not moving outside of the enclosing mesh.

The algorithm of the multigrid optimization has the following form:

1. REPEAT {
2. FOR EACH VERTEX  $x$  OF  $G_1$  ENCLOSED BY A MESH  $y_1, y_2, y_3, y_4$  OF GRID  $G_2$ :  
 $x = x - s_x \cdot \nabla F(x)$
3. FOR EACH VERTEX  $x$  OF  $G_2$  ENCLOSED BY A MESH  $y_1, y_2, y_3, y_4$  OF GRID  $G_1$ :  
 $x = x - s_x \cdot \nabla F(x)$
4. } UNTIL TERMINATION;

If all gradients fall below a threshold or if the maximal disagreement between euclidean distances and distances calculated in the preliminary step does not change between two steps of the multigrid optimization the iteration can be terminated. Figure 4 shows a resulting grid for the CIEDE2000 color distance formula.



**Figure 4.** One of the resulting grids of the multigrid optimization for the CIEDE2000 color difference formula. The corresponding starting grid is shown in Figure 2 (solid lines).

### Constructing the Transformation CLUT

To construct the transformation CLUT for planes of constant lightness we consider one starting grid (see e.g. Figure 2) and the corresponding resulting grid of the multigrid optimization (see e.g. Figure 4). Each vertex of the starting grid maps to its corresponding vertex of the resulting grid. This forms a two-dimensional CLUT where intermediate colors can be transformed by bilinear interpolation. The overall transformation is a combination of a one-dimensional CLUT for the lightness and a two-dimensional CLUT for the lightness independent coordinates. We denote the CIEDE2000 isometric transformation to the Euclidean color space by  $\mathcal{T}_{00}$  and the transformation to the Euclidean space that is isometric concerning the new optimized CIECAM02 color difference formula by  $\mathcal{T}_{02-OPT}$ . The structure of the  $\mathcal{T}_{00}$  transformation is as follows:

$$\mathcal{T}_{00} : \begin{cases} \text{CIELAB} & \mapsto \mathbb{R}^3 \\ (L^*, a^*, b^*) & \mapsto \\ (L_{00}^*(L^*), a_{00}^*(a^*, b^*), b_{00}^*(a^*, b^*)) & \end{cases} \quad (8)$$

The structure of the  $\mathcal{T}_{02-OPT}$  transformation is similar.

## Results and Discussion

For the construction of the color space transformations by means of the multigrid optimization we used two starting grids with 101 and 103 grid points in each dimension. The lattice spacings of both grids were 2.55 CIELAB units for the CIEDE2000 color difference formula and 2.55 CIECAM02 units for the optimized CIECAM02 color difference formula. To visualize the performance of the color space transformation we plotted unity color difference ellipses for both color difference formulas in the domain space. These ellipses are shown in Figure 5a for the CIEDE2000 color distance formula in CIELAB and in Figure 6a for the new optimized CIECAM02 color difference formula in the CIECAM02 color space. Then we transformed these ellipses using the CLUT transformation (see equation (8) for CIEDE2000). Figure 5b shows the transformed ellipses for the CIEDE2000 formula and Figure 6b shows the ellipses for the new optimized CIECAM02 color difference formula.

Figure 5a and 6a show that the size of the ellipses are strongly dependent on the location within the color space. In general their size increased with increasing chroma. In 5a the influence of the rotation term of the CIEDE2000 formula around hue angle of  $275^\circ$  is clearly noticeable. The ellipses are rotated here anti-clockwise. In Figures 5b and 6b the ellipses are nearly circles independent of their location within the color space. One exception is the area around hue angle of  $275^\circ$  for the CIEDE2000 formula where the ellipses obviously differ from perfect unity circles. Here, where the rotation term is active, the magnitude of the Gaussian curvature is large and an isometric transformation is not possible.

To quantify the magnitude of disagreement, we used the following disagreement formula:

$$D_{00}(x, y) = 100 \times \left( \frac{\max\{\Delta E_{00}^*(x, y), \|\mathcal{T}_{00}(x) - \mathcal{T}_{00}(y)\|_2\}}{\min\{\Delta E_{00}^*(x, y), \|\mathcal{T}_{00}(x) - \mathcal{T}_{00}(y)\|_2\}} - 1 \right) \quad (9)$$

A similar formula was used for the new optimized CIECAM02 color distance formula.

We chose 2 million randomly distributed colors within the CIELAB space and within the CIECAM02 space. For each of these colors we chose randomly a second color with a maximal difference of 5 CIELAB units or 5 CIECAM02 units, respectively. For the resulting 2 million color pairs in the CIELAB space and in the CIECAM02 space we calculated the disagreement value according to equation (9). Additionally, we calculated the PF/3 value [17]. The results are shown in Table 1.

### Magnitude of disagreement

	CIEDE2000	CIECAM02 OPT
mean	2.2% [ $D_{00}$ ]	1.2% [ $D_{02-OPT}$ ]
std	3.6% [ $D_{00}$ ]	1.6% [ $D_{02-OPT}$ ]
max	113.7% [ $D_{00}$ ]	24.6% [ $D_{02-OPT}$ ]
PF/3	2.5	1.8

In Figure 7a CIEDE2000 differences are plotted against euclidean distances after transforming the corresponding color pairs into the new space by  $\mathcal{T}_{00}$ . Figure 7b shows the same plot for

the new optimized CIECAM02 color difference formula and the  $\mathcal{T}_{02-OPT}$  color space transformation. In both figures the points are lying close to unity. In Figure 7a some outliers are visible. The corresponding colors are located around hue angle of  $275^\circ$  where the Gaussian curvature of the space is large and therefore the isometric disagreement is large as well. Here also the largest disagreement value of about 114% is located.

## Conclusion

This paper shows a computational technique of how to construct a nearly isometric transformation from a color space with a non-Euclidean color difference formula into an Euclidean space. The resulting transformation is a combination of a one-dimensional color lookup table (CLUT) to transform lightness values and a two-dimensional CLUT to transform chroma and hue coordinates. As an example the CIEDE2000 formula and a new optimized color difference formula for CIECAM02 was used and a transformation into an Euclidean space was calculated. The mean isometric disagreement was far below 3%.

Since geodesics in Euclidean spaces are straight lines, even large color distances can be easily calculated. This, can be a starting point in investigating MacAdams *diminishing returns in color distance perception* effect.

In color engineering applications that minimize color distances, the calculation can be performed in the new Euclidean space instead of dealing with a complex color difference formula like CIEDE2000.

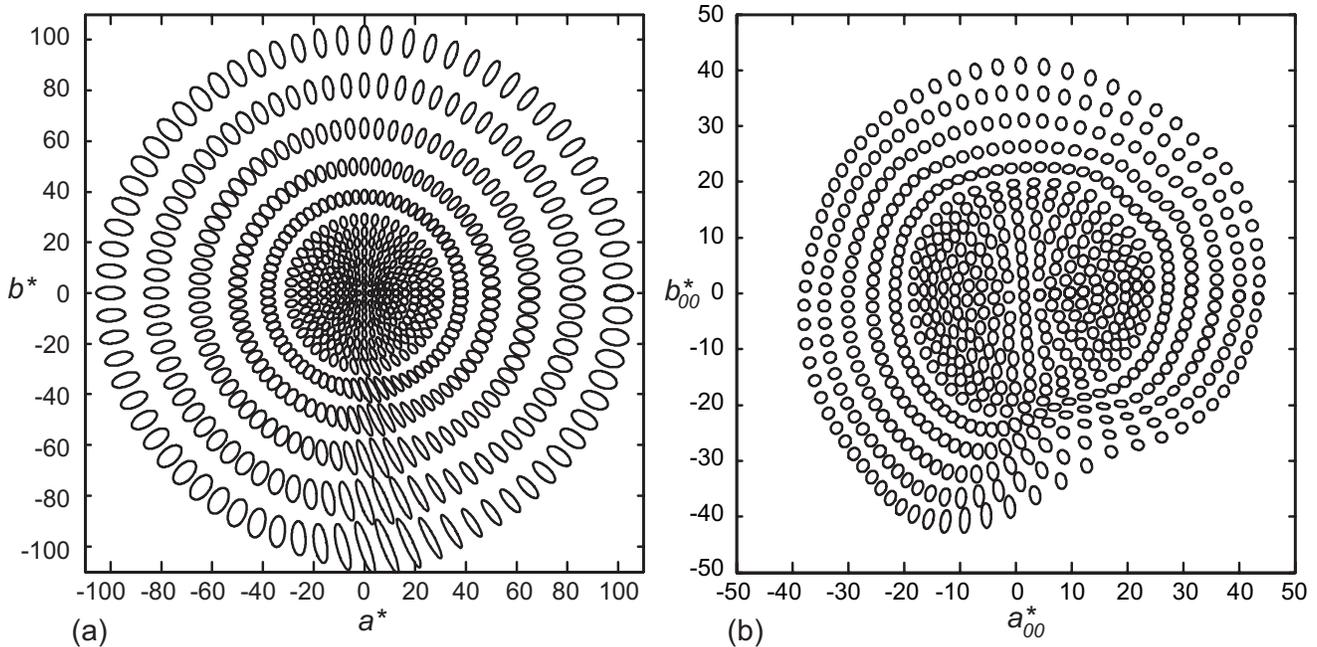
We provide a movie of how the CIEDE2000 system is embedded into an Euclidean space using the multigrid optimization on the website: <http://munsell.cis.rit.edu/~pmupci/>

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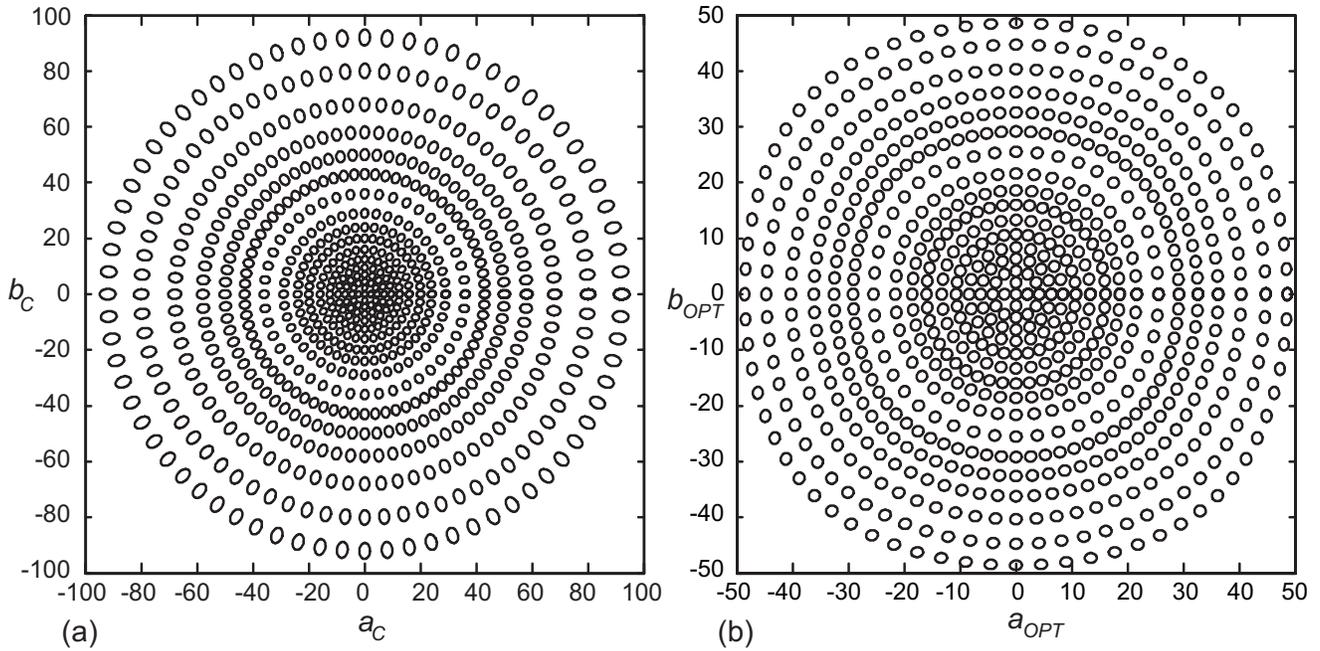
**Figure 5.** a)  $\Delta E_{00}^* = 1$  tolerance ellipsoids projected onto the  $a^*, b^*$  plane of the CIELAB color space. b) The same ellipses transformed into the new Euclidean color space

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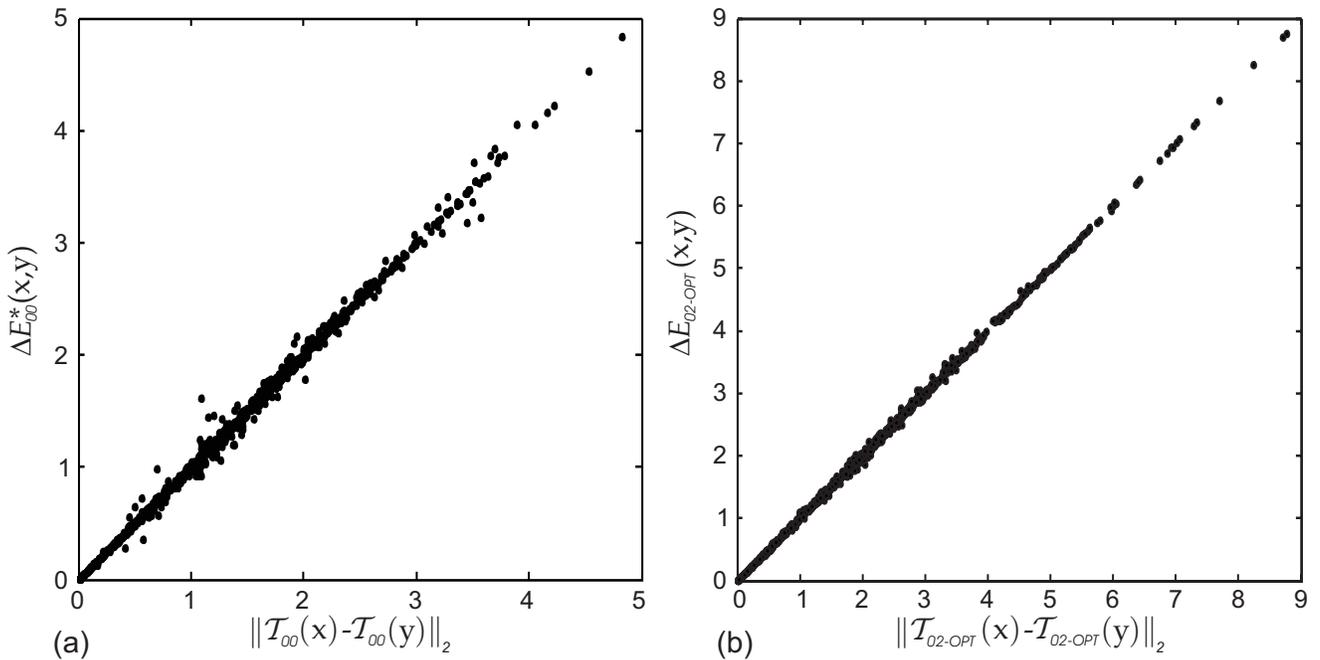
## Author Biography

Philipp Urban received his M.S. degree in Mathematics from the University of Hamburg in 1999 and his Dr. degree in the

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**Figure 6.** a)  $\Delta E_{02-0PT}^*$  tolerance ellipsoids projected onto a plane of constant lightness of the CIECAM02 color space. b) The same ellipses transformed into the new Euclidean color space



**Figure 7.** a)  $\Delta E_{00}^*$  distances plotted against Euclidean distances within the new color space derived from the CIELAB space with high isometric agreement to the CIEDE2000 formula b)  $\Delta E_{02-0PT}^*$  distances plotted against Euclidean distances within the new color space derived from the CIECAM02 color space with high isometric agreement to the new CIECAM02 color distance formula.