Visualization and Quantity Estimation of Mismatch Gamuts of Metamers viewed under different Illuminants

Philipp Urban and Rolf–Rainer Grigat
Ratio Entwicklungen GmbH, Technische Universität Hamburg-Harburg
Hamburg, Germany
Corresponding author: P. Urban (urban@tu-harburg.de)

ABSTRACT

This paper deals with the effect of illuminant metamericism. A method is proposed which characterizes the mismatch gamut of all reflective metamers for a set of illuminants which are viewed under a different illuminant. The mismatch gamut is described by a Metamer Boundary Descriptor (MBD) matrix in the CIE L*a*b* color space which is calculated in the CIE XYZ color space using linear programming. The MBD matrix can be used to visualize the mismatch gamut and to calculate it’s volume in the CIE L*a*b* color space. A closed formula to determine the volume using the MBD matrix will be presented. Results of simulation experiments for various illuminants and reflection spectra will be given.

1. INTRODUCTION

Reflective metamers are pairs of color stimuli with equal tristimulus values for a set of illuminating illuminants but with different reflection spectra. For an illuminant which is not in the set of the mentioned illuminants the reflection spectra result in different tristimulus values. To determine the theoretical magnitude of color mismatches a Monte-Carlo method [1] or linear programming [2] have been used.

In this paper we propose a method to characterize the mismatch gamut boundaries in the CIE L*a*b* color space by calculating a Metamer Boundary Descriptor (MBD) matrix. This MBD matrix approximately describes the mismatch gamut by storing its boundary points in every entry. To calculate the entries of the MBD we use a priori knowledge about the physics of natural reflectance spectra and also a linear programming technique [3].

Using the MBD the volume of the mismatch gamut can be easily calculated in the nearly perceptual uniform CIEXYZ color space.

2. METHOD

In the following text we describe the problem in a discrete formulation by sampling all spectra in N equal wavelength intervals in the range of [400 nm, 700 nm]. The tristimulus values of a reflection spectrum r for a illuminant l are given by

\[(X, Y, Z)^T = \Omega_l r\]  where  \(\Omega_l = \begin{pmatrix}
 l_1 \cdot x_1 & \cdots & l_N \cdot x_N \\
 l_1 \cdot y_1 & \cdots & l_N \cdot y_N \\
 l_1 \cdot z_1 & \cdots & l_N \cdot z_N 
\end{pmatrix}\)  and the color matching functions \(x, y, z\).

Two reflection spectra \(r_1, r_2\) (\(r_1 \neq r_2\)) are called metamer for a set of illuminants \(l_1, \ldots, l_M\) if they apply

\(\Omega_{l_i} r_1 = \Omega_{l_i} r_2 = (X_i, Y_i, Z_i)^T, \ i = 1, \ldots, M\)

For a different illuminant \(L\) they might not match, i.e.

\(\Omega_L r_1 \neq \Omega_L r_2\)

To determine all tristimulus values which occur under the illuminant \(L\) for metamers relating to the set of illuminants \(l_1, \ldots, l_M\) we calculate the boundary of the mismatch gamut using a Metamer Boundary Descriptor (MBD).
In the following we denote a mismatch gamut in the CIE XYZ color space by $\textit{XYZM}$ and the corresponding gamut in CIE L*a*b* color space by $\textit{LabM}$ (we omit the symbols of the participate illuminants for clarity). The transformation from CIE XYZ to CIE L*a*b* is denoted by $L$, so $L(\textit{XYZM}) = \textit{LabM}$ applies.

The MBD is a $n \times m$ matrix which stores a boundary point of the mismatch gamut $\textit{LabM}$ in every entry (see Figure 1a). Each row contains $m$ contour points of the set for a fixed L* value. The boundary points will be calculated in the CIEXYZ space using the following linear programming (LP) problem, which samples the metamer subspace along a straight line $g + \lambda \cdot v$:

Solve: 

$$\lambda = \min$$

using the linear constraints: $r \geq 0$, $r \leq 1$, $Hr \leq p$, $-Hr \leq p$, $\Omega_i \cdot r = (X_i, Y_i, Z_i)^T$, $i = 1, \ldots, M$, $\Omega_i \cdot r = v_r$, $g + \lambda v = v_r$, $\lambda \geq 0$

Constraint $r \geq 0$ ensures the positivity of reflectance spectra and constraint $r \leq 1$ guarantees the boundness for non-fluorescent surfaces. In addition, we use the smoothness constraint $Hr \leq p$ and $-Hr \leq p$ with a smoothing parameter $p > 0$ and a convolution matrix $H$ to apply the Laplace operator to $r$. Constraint $\Omega_i \cdot r = (X_i, Y_i, Z_i)^T$, $i = 1, \ldots, M$ ensures that we only take into account reflectance spectra $r$ which are metamers relating to the set of illuminants. Constraint $\Omega_i \cdot r = v_r$ introduces the considered illuminant $L$ with the auxiliary variable $v_r$ which is necessary for the sampling constraint $g + \lambda v = v_r$. This constraint together with the objective function ($\lambda = \min$) allows us to sample the boundary of the mismatch gamut in the CIE XYZ space along the mentioned line (see Figure 1b). After solving this LP we get a tristimulus value $v_r$, which is the intersection of the line and the boundary of the mismatch gamut and the appropriate reflection spectrum $r$ as results. For each boundary point the parameters $g$ (anchor point) and $v$ (direction vector) defining the sampling line have to be chosen in a way that the MBD entries are uniformly distributed on the boundary of the mismatch gamut in the CIE L*a*b* color space. A special technique for choosing these parameters is explained in [3].

The MBD matrix represents a wired grid model of the corresponding mismatch gamut, so it is suitable for visualization, e.g. using Matlab [4] (see Figure 2).

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**Figure 1:** a) The MBD matrix with $n = 5$ and $m = 8$ b). The corresponding metamer subspace $M_{\textit{XYZ}}$ in CIE XYZ color space. Each entry of the MBD matrix is sampled in the CIE XYZ color space using a linear programming technique. The sampled point $v_r$ is transformed in the CIE L*a*b* color space and stored in the MBD matrix.
3. VOLUME OF THE MISMATCH GAMUT

The volume of a mismatch gamut $M_{Lab}$ in the CIE L*a*b* color space is given by

$$Vol(M_{Lab}) = \int_{M_{Lab}} dL' da' db'$$

Using different substitutions the integral above can be transformed in

$$Vol(M_{Lab}) = \int_{Y_{min}}^{Y_{max}} \int_{R_{min}}^{R_{max}} \int_{\phi_{min}}^{\phi_{max}} |\det[DL(X(R \cos(\phi), Y), Y, Z(Y, R \sin(\phi)))]| dR d\phi dY$$

where $L(X,Y,Z)$ is the transformation from CIE XYZ to CIE L*a*b*, $DL$ is the corresponding Jacobian matrix, $X(X_0,Y), Z(Z_0,Y)$ translate the origin to a point on the anchor line (see Figure 1b), $Y_{min}$, $Y_{max}$ are the minimal and the maximal $Y$ values of $M_{XYZ}$ and $R(Y, \phi)$ is a function which determines the distance of the boundary point of $M_{XYZ}$ from the anchor point. $R(Y, \phi)$ is implicitly given for the entries of the MBD matrix end can be determined in the following manner:

Every MBD matrix entry can be written as follows in the CIE XYZ color space

$$a_{i,j} = \begin{bmatrix}
X_{i,j} \\
Y_{i,j} \\
Z_{i,j}
\end{bmatrix} = \begin{bmatrix}
R_{i,j} \cos(\phi_{i,j}) \\
0 \\
R_{i,j} \sin(\phi_{i,j})
\end{bmatrix} + g_i$$

where $g_i = (g_{i,x}, g_{i,y}, g_{i,z})^T$ is the anchor point of all sampling lines related to the $i$th row.

The $Y_i := g_{i,y} \phi_{i,j}$ and $R_{i,j}$ describe sheared cylindrical coordinates and can be stored in a matrix

$$\begin{bmatrix}
(Y_1, \phi_{1,1}, R_{1,1})^T \\
\vdots \\
(Y_n, \phi_{n,1}, R_{n,1})^T
\end{bmatrix} \cdots \begin{bmatrix}
(Y_1, \phi_{1,m}, R_{1,m})^T \\
\vdots \\
(Y_n, \phi_{n,m}, R_{n,m})^T
\end{bmatrix}$$

Using a decomposition $0 = R_{i,j,k} < \cdots < R_{i,j,K} = R_{i,j} = R(Y_i, \phi_{i,j})$ we can approximate the integral by a sum

$$Vol(M_{Lab}) \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{K} R_{i,j,k} |\det[DL(X(R_{i,j,k} \cos(\phi_{i,j}), Y), Y, Z(Y, R_{i,j,k} \sin(\phi_{i,j})))]| \Delta R_{i,j,k} \Delta \phi_{i,j} \Delta Y$$

If the MBD matrix has been calculated initially the above sum describes a closed formula to calculate the volume of the mismatch gamut.

![Figure 2: Mismatch gamuts of metamer to sample 10 for illuminant CIE C (left) and CIE C, CIE A (right) viewed under illuminant CIE F2.](image)

4. RESULTS

We have calculated the volumes of various mismatch gamuts of metamer which match to the 24 reflection spectra of a MacBeth Color Checker under one or two standard CIE illuminants and are
viewed under a different CIE illuminant. The involved CIE illuminants are CIE A (incandescent light), CIE C (average daylight,) and CIE F2 (cool white fluorescent). The MBD matrix dimension has been set to 8×8. All spectra have been sampled in 5 nm wavelength intervals. The smoothness parameter $p$ has been set to 0.0035. The results are presented in Table 1.

Table 1: Volumes of metamer mismatch gamuts in the CIE L*a*b* color space for metamers which match the reflection spectra of the MacBeth Color Checker for one or two illuminants and are viewed under an different illuminant.

<table>
<thead>
<tr>
<th>Viewing illum.</th>
<th>A C, F2</th>
<th>C</th>
<th>C A, F2</th>
<th>F2</th>
<th>A A, C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 1</td>
<td>22.5140</td>
<td>0.1527</td>
<td>27.8000</td>
<td>0.1265</td>
<td>1012.0000</td>
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<tr>
<td>Sample 2</td>
<td>1.9837</td>
<td>0.0000</td>
<td>2.9828</td>
<td>0.0000</td>
<td>105.9200</td>
</tr>
<tr>
<td>Sample 3</td>
<td>6.2987</td>
<td>0.0427</td>
<td>5.9226</td>
<td>0.0242</td>
<td>279.9000</td>
</tr>
<tr>
<td>Sample 4</td>
<td>15.8150</td>
<td>0.1342</td>
<td>18.2280</td>
<td>0.0859</td>
<td>610.8300</td>
</tr>
<tr>
<td>Sample 5</td>
<td>3.3351</td>
<td>0.0140</td>
<td>4.0462</td>
<td>0.0077</td>
<td>155.7000</td>
</tr>
<tr>
<td>Sample 6</td>
<td>0.6674</td>
<td>0.0043</td>
<td>0.8821</td>
<td>0.0028</td>
<td>35.2140</td>
</tr>
<tr>
<td>Sample 7</td>
<td>2.3543</td>
<td>0.0000</td>
<td>3.4754</td>
<td>0.0000</td>
<td>127.3200</td>
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<tr>
<td>Sample 8</td>
<td>10.3250</td>
<td>0.0456</td>
<td>7.7043</td>
<td>0.0181</td>
<td>425.5300</td>
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<tr>
<td>Sample 9</td>
<td>3.0434</td>
<td>0.0000</td>
<td>3.4987</td>
<td>0.0000</td>
<td>119.1400</td>
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<td>Sample 10</td>
<td>30.3760</td>
<td>0.2204</td>
<td>31.9710</td>
<td>0.1538</td>
<td>1534.2000</td>
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<tr>
<td>Sample 11</td>
<td>0.5679</td>
<td>0.0000</td>
<td>0.6110</td>
<td>0.0000</td>
<td>13.2230</td>
</tr>
<tr>
<td>Sample 12</td>
<td>1.5618</td>
<td>0.0087</td>
<td>2.1819</td>
<td>0.0078</td>
<td>85.5670</td>
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<tr>
<td>Sample 13</td>
<td>17.8910</td>
<td>0.0122</td>
<td>11.4620</td>
<td>0.0150</td>
<td>649.5600</td>
</tr>
<tr>
<td>Sample 14</td>
<td>2.2879</td>
<td>0.0000</td>
<td>3.5323</td>
<td>0.0000</td>
<td>115.7700</td>
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<td>Sample 15</td>
<td>5.6815</td>
<td>0.0000</td>
<td>5.7598</td>
<td>0.0000</td>
<td>160.6700</td>
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<tr>
<td>Sample 16</td>
<td>0.4156</td>
<td>0.0000</td>
<td>0.6436</td>
<td>0.0000</td>
<td>15.9410</td>
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<tr>
<td>Sample 17</td>
<td>1.5681</td>
<td>0.0000</td>
<td>1.6485</td>
<td>0.0000</td>
<td>52.7540</td>
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<tr>
<td>Sample 18</td>
<td>3.3320</td>
<td>0.0118</td>
<td>3.2222</td>
<td>0.0037</td>
<td>164.0000</td>
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<td>Sample 19</td>
<td>0.1933</td>
<td>0.0014</td>
<td>19.5640</td>
<td>1.2764</td>
<td>8.9936</td>
</tr>
<tr>
<td>Sample 20</td>
<td>0.7579</td>
<td>0.0050</td>
<td>1.1395</td>
<td>0.0036</td>
<td>40.6350</td>
</tr>
<tr>
<td>Sample 21</td>
<td>2.0089</td>
<td>0.0135</td>
<td>2.9332</td>
<td>0.0098</td>
<td>108.3600</td>
</tr>
<tr>
<td>Sample 22</td>
<td>6.6028</td>
<td>0.0447</td>
<td>8.2961</td>
<td>0.0305</td>
<td>339.1000</td>
</tr>
<tr>
<td>Sample 23</td>
<td>27.0170</td>
<td>0.1932</td>
<td>27.7110</td>
<td>0.1248</td>
<td>1217.6000</td>
</tr>
<tr>
<td>Sample 24</td>
<td>64.9590</td>
<td>1.3170</td>
<td>61.5000</td>
<td>0.8674</td>
<td>2385.9000</td>
</tr>
</tbody>
</table>

5. CONCLUSIONS

We presented a method which characterises the mismatch gamuts in the CIE L*a*b* color space of metamers for a set of illuminants which are viewed under a different illuminant. The method uses a linear programming technique and a priori knowledge about natural reflection spectra to sample the boundary of the mismatch gamut and to store the sampling colors which are transformed in the CIE L*a*b* color space in a metamer boundary descriptor matrix. This matrix can be used to visualise the mismatch gamut and to approximate its volume.

References